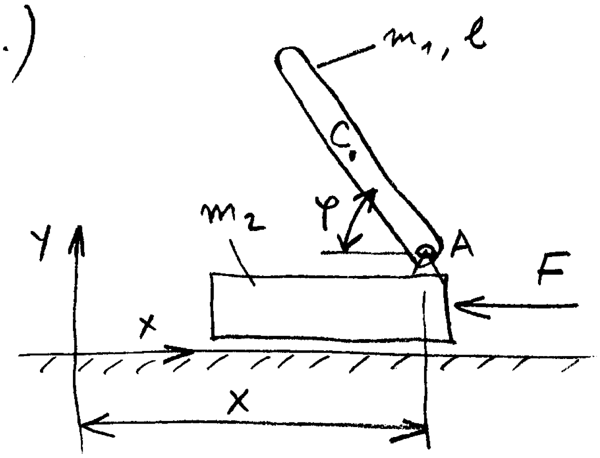
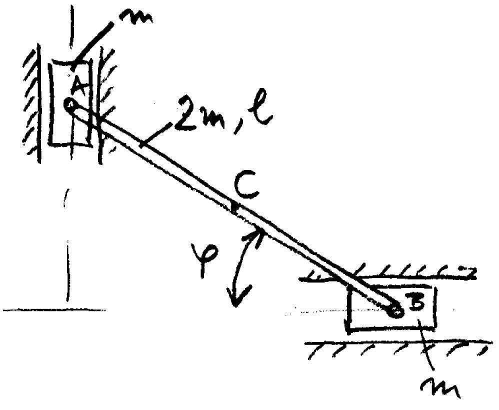
Писмен испит по Механика II   Јуни 2010

1.)



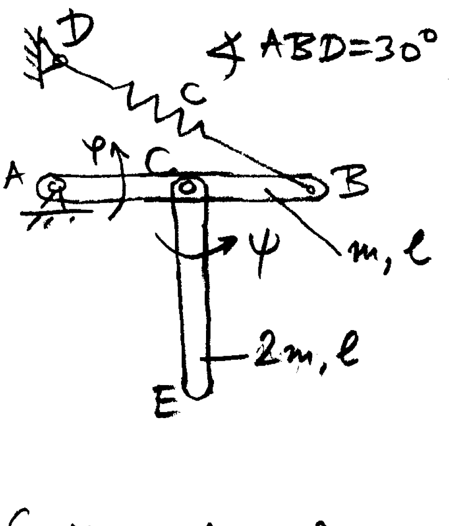
За се постават диференц. равенки на движењето на дадениот систем по генерализ. координати X и φ. Оската A е хоризонтална, а триењето да се занемари (25 б)

2.)



За се најде диференцијалната равенка на движењето на дадениот систем. За се најде аголното забрзување $\ddot{\varphi}$ ако системот се пушти да паѓа без почетна брзина од положбата $\varphi = 30^\circ$ (25 б)

3.)



За се најдат сопствените фреквенции на малите слободни непригушени осцилации на системот прикажан на скицата. Системот осцилира во вертикалната рамнина (25 б.)

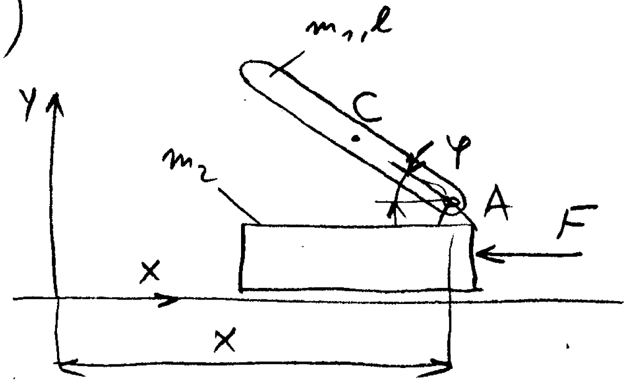
4.)

$$\begin{cases} 2\ddot{x}_1 + 4x_1 + 2x_2 = 0 \\ 3\ddot{x}_2 + 2x_1 + x_2 = 0 \end{cases}$$

$$\begin{cases} x_1(t=0) = 1 & \dot{x}_1(t=0) = 0 \\ x_2(t=0) = -1 & \dot{x}_2(t=0) = 0 \end{cases}$$

За се најде решението на системот од диф. равенки на слободните непригушени осцилации на механ. систем за дадените почетни услови (25 б)

1)



$$E_K = \frac{1}{2} m_1 \cdot v_c^2 + \frac{1}{2} J_c \cdot \dot\varphi^2 + \frac{1}{2} m_2 \dot x^2$$

$$\begin{cases} x_c = x - \frac{\ell}{2}\cos\varphi & \dot x_c = \dot x + \frac{\ell}{2}\sin\varphi\cdot\dot\varphi \\ y_c = \frac{\ell}{2}\cdot\sin\varphi & \dot y_c = \frac{\ell}{2}\cos\varphi\cdot\dot\varphi \end{cases}$$

$$v_c^2 = \dot x_c^2 + \dot y_c^2 = \left(\dot x + \frac{\ell}{2}\sin\varphi\cdot\dot\varphi\right)^2 + \left(\frac{\ell}{2}\cos\varphi\cdot\dot\varphi\right)^2$$

$$v_c^2 = \dot x^2 + \ell\sin\varphi\cdot\dot x\cdot\dot\varphi + \frac{\ell^2}{4}\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}\cdot\dot\varphi^2 = \dot x^2 + \ell\sin\varphi\cdot\dot x\cdot\dot\varphi + \frac{\ell^2}{4}\dot\varphi^2$$

$$E_K = \frac{1}{2} m_1 \underbrace{\left(\dot x^2 + \ell\sin\varphi\cdot\dot x\dot\varphi + \frac{\ell^2}{4}\dot\varphi^2\right)}_{v_c^2} + \frac{1}{2}\cdot\underbrace{\frac{1}{12} m_1 \ell^2}_{J_c}\cdot\dot\varphi^2 + \frac{1}{2} m_2 \dot x^2$$

$$\boxed{\begin{array}{l} E_K = \frac{1}{2}(m_1+m_2)\dot x^2 + \frac{1}{2} m_1 \ell\sin\varphi\cdot\dot x\dot\varphi + \frac{1}{6} m_1 \ell^2\dot\varphi^2 \\ E_p = m_1 g\, y_c = m_1 g \frac{\ell}{2}\sin\varphi \quad ; \begin{cases} \frac{d}{dt}\frac{\partial E_K}{\partial \dot x} - \frac{\partial E_K}{\partial x} + \frac{\partial E_p}{\partial x} = Q_x \\ \frac{d}{dt}\frac{\partial E_K}{\partial \dot\varphi} - \frac{\partial E_K}{\partial \varphi} + \frac{\partial E_p}{\partial \varphi} = Q_\varphi \end{cases} \\ Q_x = -F, \quad Q_\varphi = 0 \end{array}}$$

$$\frac{\partial E_K}{\partial \dot x} = (m_1+m_2)\dot x + \frac{1}{2} m_1 \ell\sin\varphi\cdot\dot\varphi \qquad \frac{\partial E_K}{\partial \dot\varphi} = \frac{1}{2} m_1 \ell\sin\varphi\cdot\dot x + \frac{1}{3} m_1 \ell^2\dot\varphi$$

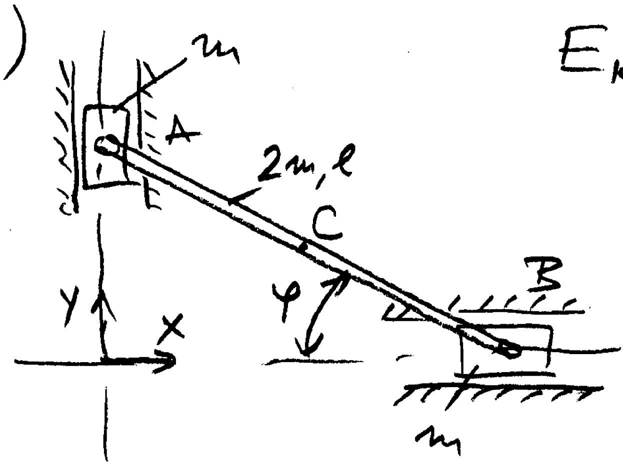
$$\frac{d}{dt}\frac{\partial E_K}{\partial \dot x} = (m_1+m_2)\ddot x + \frac{1}{2} m_1 \ell\sin\varphi\cdot\ddot\varphi + \frac{1}{2} m_1 \ell\cos\varphi\cdot\dot\varphi^2$$

$$\frac{d}{dt}\frac{\partial E_K}{\partial \dot\varphi} = \frac{1}{2} m_1 \ell\sin\varphi\cdot\ddot x + \frac{1}{2} m_1 \ell\cos\varphi\cdot\dot x\cdot\dot\varphi + \frac{1}{3} m_1 \ell^2\ddot\varphi$$

$$\frac{\partial E_K}{\partial x} = 0, \quad \frac{\partial E_K}{\partial \varphi} = \frac{1}{2} m_1 \ell\cos\varphi\cdot\dot x\dot\varphi, \quad \frac{\partial E_p}{\partial x} = 0, \quad \frac{\partial E_p}{\partial \varphi} = m_1 g \frac{\ell}{2}\cos\varphi$$

$$\boxed{\begin{cases} (m_1+m_2)\ddot x + \frac{1}{2} m_1 \ell\sin\varphi\cdot\ddot\varphi + \frac{1}{2} m_1 \ell\cos\varphi\cdot\dot\varphi^2 = -F \\ \frac{1}{2} m_1 \ell\sin\varphi\cdot\ddot x + \frac{1}{3} m_1 \ell^2\ddot\varphi + m_1 g\frac{\ell}{2}\cos\varphi = 0 \end{cases}}$$

2.)



2

$$E_K = \frac{1}{2} m v_A^2 + \frac{1}{2} m v_B^2 +$$

$$+ \cancel{\frac{1}{2}} 2m\, v_C^2 + \frac{1}{2} J_C \cdot \dot\varphi^2$$

$$y_A = l \cdot \sin\varphi \,, \quad \dot y_A = v_A = l\cos\varphi \cdot \dot\varphi$$

$$x_B = l \cdot \cos\varphi \,, \quad \dot x_B = v_B = -l\sin\varphi \cdot \dot\varphi$$

$$x_C = \frac{x_B}{2} = \frac{l}{2}\cos\varphi \qquad y_C = \frac{y_A}{2} = \frac{l}{2}\sin\varphi$$

$$\dot x_C = -\frac{l}{2}\sin\varphi \cdot \dot\varphi \qquad \dot y_C = \frac{l}{2}\cos \cdot \dot\varphi \qquad v_C^2 = \dot x_C^2 + \dot y_C^2 = \frac{l^2}{4} \cdot \dot\varphi^2$$

$$E_K = \frac{1}{2} m \underbrace{l^2 \cos^2\varphi \cdot \dot\varphi^2}_{v_A^2} + \frac{1}{2} m \underbrace{\cdot\, l^2 \sin^2\varphi \cdot \dot\varphi^2}_{v_B^2} + m \cdot \underbrace{\frac{l^2}{4} \cdot \dot\varphi^2}_{v_C^2} + \cancel{\frac{1}{2}} \cdot \underbrace{\frac{1}{12} \cancel{2} m l^2}_{J_C} \dot\varphi^2$$

$$\boxed{E_K = \frac{5}{6} m l^2 \dot\varphi^2}$$

$$E_p = mg\, y_A + 2mg\, y_C = mg\, l \sin\varphi + \cancel{2} mg \frac{l}{\cancel{2}} \sin\varphi =$$

$$\boxed{E_p = 2mgl \sin\varphi}$$

$$\frac{d}{dt}\frac{\partial E_K}{\partial \dot\varphi} - \cancelto{0}{\frac{\partial E_K}{\partial \varphi}} + \frac{\partial E_p}{\partial \varphi} = \cancelto{0}{Q_\varphi}$$

$$\frac{5}{3} m l^2 \cdot \ddot\varphi + 2mgl\cos\varphi = 0 \quad /: \frac{5}{3} m l^2$$

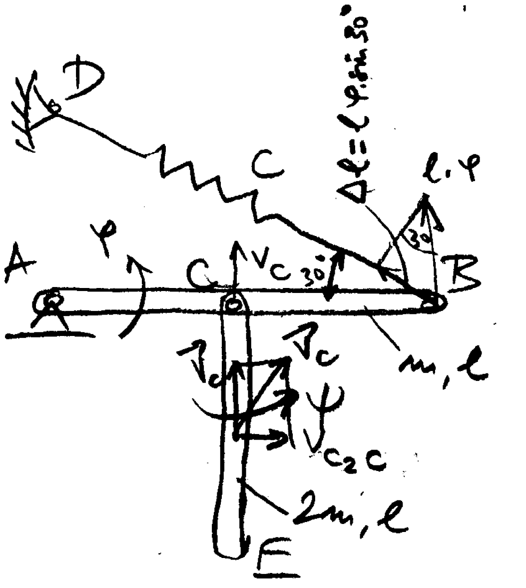
$$\boxed{\ddot\varphi = -\frac{6}{5} \cdot \frac{g}{l} \cos\varphi}$$

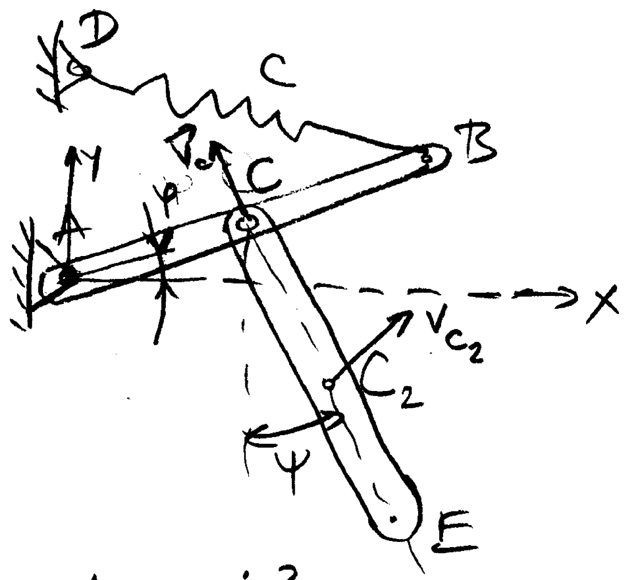
За $\varphi = 30°$

$\cos\varphi = \frac{\sqrt{3}}{2}$

$$\boxed{\ddot\varphi = -\frac{3\sqrt{3}}{5} \cdot \frac{g}{l}}$$

3.)

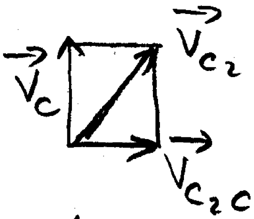




3

$$E_K=\frac{1}{2}J_A\cdot\dot\varphi^2+\frac{1}{2}\cdot 2m\cdot v_{C_2}^2+\frac{1}{2}J_{C_2}\cdot\dot\psi^2$$

$$J_A=\frac{1}{3}ml^2\ ,\ J_{C_2}=\frac{1}{12}\cdot 2ml^2\ ,$$



За $\varphi=\psi=0$ $\quad v_{C_2}^2=v_C^2+v_{C_2C}^2 \quad v_C=\frac{l}{2}\dot\varphi \quad v_{C_2C}=\frac{l}{2}\dot\psi$
(мали осцил.)

$$v_{C_2}^2=\left(\frac{l}{2}\dot\varphi\right)^2+\left(\frac{l}{2}\dot\psi\right)^2=\frac{l^2}{4}\left(\dot\varphi^2+\dot\psi^2\right)$$

$$E_K=\frac{1}{2}\cdot\frac{1}{3}ml^2\dot\varphi^2+\frac{1}{\cancel{2}}\cancel{2}m\frac{l^2}{4}\left(\dot\varphi^2+\dot\psi^2\right)+\frac{1}{\cancel{2}}\cdot\frac{1}{12}\cancel{2}ml^2\dot\psi^2$$

$$\boxed{E_K=\frac{7}{12}ml^2\dot\varphi^2+\frac{1}{3}ml^2\dot\psi^2}$$

$$\Delta l=l\cdot\varphi\cdot\sin 30^\circ=\frac{l\varphi}{2}$$

$$E_p=mg\,y_C+2mg\,y_{C_2}+\frac{1}{2}c\left(\Delta l_{st}-\Delta l\right)^2-\frac{1}{2}c\,\Delta l_{st}^2=$$

$$=mg\frac{l}{2}\underbrace{\sin\varphi}_{\approx\varphi}+\cancel{2}mg\left(\frac{l}{\cancel{2}}\underbrace{\sin\varphi}_{\approx\varphi}+\frac{l}{\cancel{2}}\underbrace{(1-\cos\psi)}_{\approx\psi^2/2}\right)+\frac{1}{2}c\left(\Delta l_{st}-\frac{l\varphi}{2}\right)^2-\frac{1}{2}c\Delta l_{st}^2$$

$$E_p=\frac{3}{2}mgl\varphi+\frac{1}{2}mgl\psi^2+\cancel{\frac{1}{2}c\Delta l_{st}^2}-c\Delta l_{st}\cdot\frac{l\varphi}{2}+\frac{1}{8}cl^2\varphi^2-\cancel{\frac{1}{2}c\Delta l_{st}^2}$$

$$E_p=\frac{1}{8}cl^2\varphi^2+\frac{1}{2}mgl\psi^2+\underbrace{\left(\frac{3}{2}mgl-c\Delta l_{st}\frac{l}{2}\right)}_{0}\varphi$$

$$\left.\frac{\partial E_p}{\partial\varphi}\right|_{\varphi,\psi=0}=0:\quad \left.\frac{\partial E_p}{\partial\varphi}\right|_{\varphi,\psi=0}=\frac{1}{4}cl^2\underbrace{\cancel{\varphi}}_{0}+mgl\underbrace{\cancel{\psi}}_{0}+\boxed{\frac{3}{2}mgl-c\Delta l_{st}\frac{l}{2}=0}$$

$$\boxed{E_p=\frac{1}{8}cl^2\varphi^2+\frac{1}{2}mgl\psi^2}$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial E_k}{\partial \dot\varphi} - \dfrac{\partial E_k}{\partial \varphi} + \dfrac{\partial E_p}{\partial \varphi} = 0 \\ \dfrac{d}{dt}\dfrac{\partial E_k}{\partial \dot\psi} - \dfrac{\partial E_k}{\partial \psi} + \dfrac{\partial E_p}{\partial \psi} = 0 \end{cases} \qquad \begin{cases} \dfrac{5}{6} m \cancel{l^2} \ddot\varphi + \dfrac{1}{4} c \cancel{l^2} \varphi = 0 \\ \dfrac{2}{3} \cancel{m} l^{\cancel{2}} \ddot\psi + \cancel{m} g \cancel{l} \psi = 0 \end{cases}$$

4

$\varphi(t) = A\cos(\omega t - \alpha)$, $\psi(t) = B\cos(\omega t - \alpha)$

$$\begin{cases} A\left(\dfrac{c}{4} - \dfrac{5}{6} m\omega^2\right) = 0 \\ B\left(g - \dfrac{2}{3} l\omega^2\right) = 0 \end{cases}$$

$$\Delta(\omega^2) = \begin{vmatrix} \dfrac{c}{4} - \dfrac{5}{6} m\omega^2 & 0 \\ 0 & g - \dfrac{2}{3} l\omega^2 \end{vmatrix} = \left(\dfrac{c}{4} - \dfrac{5}{6} m\omega^2\right)\cdot\left(g - \dfrac{2}{3} l\omega^2\right) = 0$$

$$\rightarrow \boxed{\omega_1 = \sqrt{\dfrac{3}{10}\dfrac{c}{m}}, \quad \omega_2 = \sqrt{\dfrac{2}{3}\cdot\dfrac{g}{l}}}$$

4.) $$\begin{cases} 2\ddot x_1 + 4x_1 + 2x_2 = 0 \\ 3\ddot x_2 + 2x_1 + x_2 = 0 \end{cases}$$

$x_1(t) = A\cos(\omega t - \alpha)$ $\quad x_2(t) = B\cos(\omega t - \alpha)$

$$\begin{cases} (4 - 2\omega^2)A + 2B = 0 \\ 2A + (1 - 3\omega^2)B = 0 \end{cases}$$

$$\Delta(\omega^2) = \begin{vmatrix} 4-2\omega^2 & 2 \\ 2 & 1-3\omega^2 \end{vmatrix} = 0 \qquad \Delta(\omega^2) = (4-2\omega^2)(1-3\omega^2) - 4 = 0$$

$$\Delta(\omega^2) = 6(\omega^2)^2 - 5\omega^2 = 0 \qquad \omega^2_{1,2} = \frac{5 \pm \sqrt{25}}{12} \qquad \boxed{\begin{matrix}\omega_1 = 0 \\ \omega_2 = \sqrt{5/6}\end{matrix}}$$

$$\eta = \frac{B}{A} = -\frac{4-2\omega^2}{2} = -2 + \omega^2 \qquad \eta_1 = \eta(\omega_1) = -2 \qquad \eta_2 = \eta(\omega_2) = -2 + \frac{5}{6} = -\frac{7}{6}$$

Случај на еден корен еднаков на нула:

$$\begin{cases} x_1(t) = A_1 t + A_2 + A_3\cos(\omega t - \alpha) \\ x_2(t) = \eta_1(A_1 t + A_2) + \eta_2 A_3\cos(\omega t - \alpha) = -2A_1 t - 2A_2 - \frac{7}{6} A_3\cos(\omega t - \alpha) \end{cases}$$

Поч. услови:
$$\begin{cases} x_1(0) = A_1\cdot 0 + A_2 + A_3\cos(\omega 0 - \alpha) = 1 \\ x_2(0) = -2A_1\cdot 0 - 2A_2 - \frac{7}{6}A_3(\omega\cdot 0 - \alpha) = -1 \\ \dot x_1(0) = A_1 - A_3\omega\cdot\sin(\omega\cdot 0 - \alpha) = A_1 + A_3\omega\sin\alpha = 0 \quad (3) \\ \dot x_2(0) = -2A_1 + \frac{7}{6}A_3\omega\cdot\sin(\omega\cdot 0 - \alpha) = -2A_1 - \frac{7}{6}A_3\omega\sin\alpha = 0 \quad (4) \end{cases}$$

$$\begin{cases} A_2 + A_3\cos\alpha = 1 & (1) \\ -2A_2 - \frac{7}{6}A_3\cos\alpha = -1 & (2) \end{cases}$$

Од (1) и (2):

$$\boxed{A_2 = \frac{-\frac{7}{6}\cos\alpha + \cos\alpha}{-\frac{7}{6}\cos\alpha + 2\cos\alpha} = -\frac{1}{5}} \qquad A_3 = \frac{-1+2}{-\frac{7}{6}\cos\alpha + 2\cos\alpha} = \frac{6}{5}\cdot\frac{1}{\cos\alpha}$$

Од (3) и (4):
$$\begin{cases} A_1 + \frac{6}{5}\frac{\sin\alpha}{\cos\alpha}\cdot\omega = 0 \\ -2A_1 - \frac{7}{\cancel{6}}\frac{\cancel{6}}{5}\frac{\sin\alpha}{\cos\alpha}\cdot\omega = 0 \end{cases} \rightarrow \boxed{A_1 = 0}, \; tg\,\alpha = 0 \rightarrow \boxed{\alpha = 0}, \; \cos\alpha = 1 \qquad \boxed{A_3 = \frac{6}{5}}$$

$$\boxed{x_1(t) = -\frac{1}{5} + \frac{6}{5}\cos\left(\sqrt{\frac{5}{6}}t\right), \quad x_2(t) = \frac{2}{5} - \frac{7}{5}\cos\left(\sqrt{\frac{5}{6}}t\right)}$$