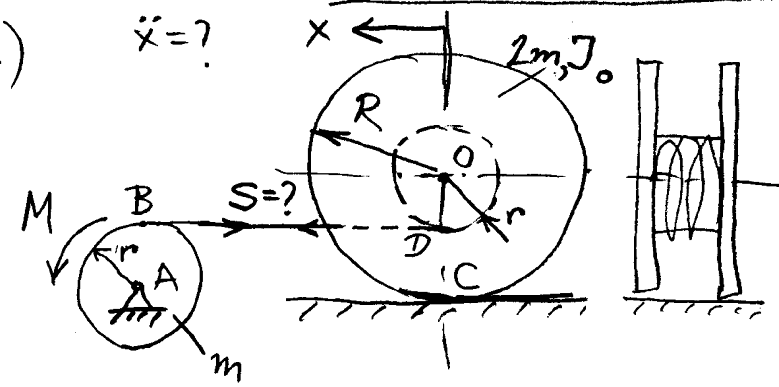
Динамика и осцилации

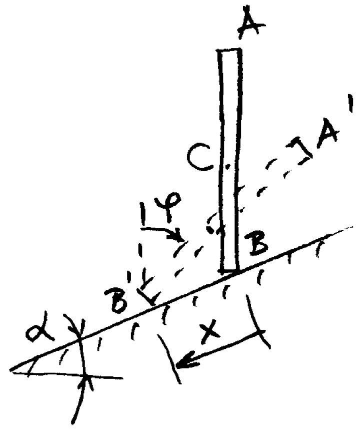
Јуни 2011
писм. исп.

1.)



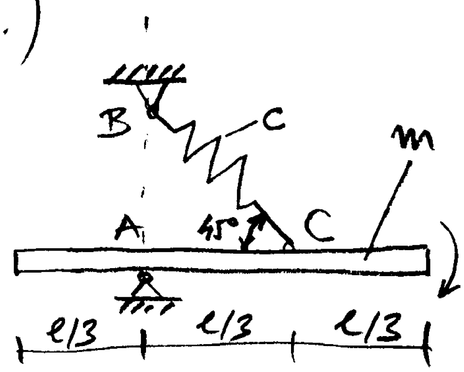
Да се најде забрзувањето $\ddot{x}$ на центарот O на макарата, ако на барабанот AB дејствува спрег M. Потоа да се најде силата S во конецот BD. (25 б.)

2.)



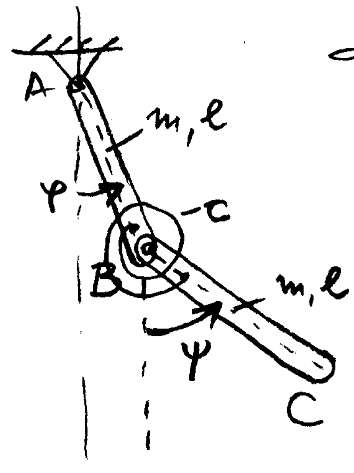
Стапот AB, со должина l, во моментот $t=0$ е оставен во вертикална положба и без почетна брзина на идеално мазна коса рамнина која зафаќа агол α со хоризонталата. Да се постават диференцијалните равенки на движењето по x и φ во вертикалната рамнина. (25 б.)

3.)



Да се определи сопствената фреквенција на малите осцилации на дадениот систем. Во рамнотежната положба $\sphericalangle ACB = 45°$. Масата на стапот е m, а крутоста на пружината е c. (25 б.)

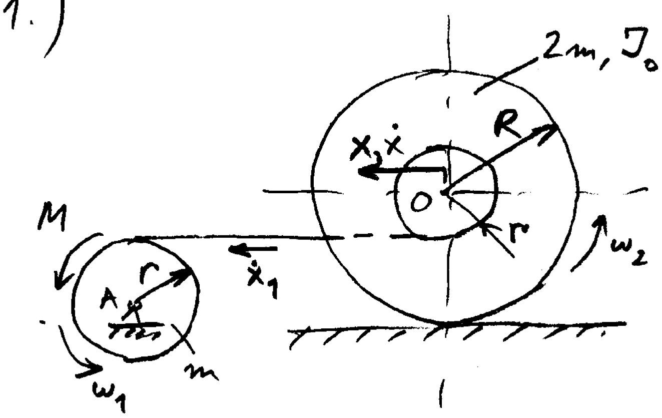
4.)



Да се постави фреквенционата равенка на даденото двојно физичко нишало со пружина во зглобот B. Во рамнотежна положба, при $\varphi=\psi=0$, пружината е ненапрегната (25 б.)

Динамика и осцилации Јуни 2011 1

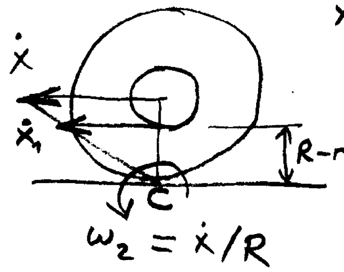
1.)



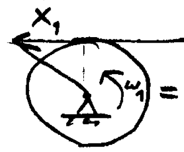
1 ст. на сл., x – генер. коорд.

$$\dot{x}_1 = \omega_2 \cdot (R-r)$$

$$\dot{x}_1 = \dot{x}\,\frac{R-r}{R}$$



$$\omega_2 = \dot{x}/R$$



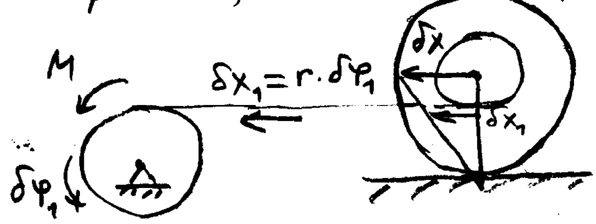
$$\omega_1 = \dot{x}_1/r$$

$$\omega_1 = \frac{\dot{x}}{r}\cdot\frac{R-r}{R}$$

$$E_K = \frac{1}{2}J_A\cdot\omega_1^2 + \frac{1}{2}\cdot 2m\cdot\dot{x}^2 + \frac{1}{2}J_0\cdot\omega_2^2 =$$

$$= \frac{1}{2}\cdot\underbrace{\frac{1}{2}m\cancel{r^2}}_{J_A}\cdot\underbrace{\left(\frac{\dot{x}}{\cancel{r}}\cdot\frac{R-r}{R}\right)^2}_{\omega_1^2} + m\dot{x}^2 + \frac{1}{2}J_0\cdot\underbrace{\left(\frac{\dot{x}}{R}\right)^2}_{\omega_2^2} = \left[\frac{1}{4}m\left(\frac{R-r}{R}\right)^2 + m + \frac{1}{2}\frac{J_0}{R^2}\right]\cdot\dot{x}^2$$

$$E_p = 0\ ;\quad \delta A = M\cdot\delta\varphi_1 = Q_x\cdot\delta x$$



$$\delta x = \frac{R}{R-r}\,\delta x_1 = \frac{R}{R-r}\cdot r\cdot\delta\varphi_1$$

$$\frac{\delta x}{R} = \frac{\delta x_1}{R-r}$$

$$\delta A = M\cdot\cancel{\delta\varphi_1} = Q_x\cdot\underbrace{\frac{R}{R-r}\cdot r\cdot\cancel{\delta\varphi_1}}_{\delta x}$$

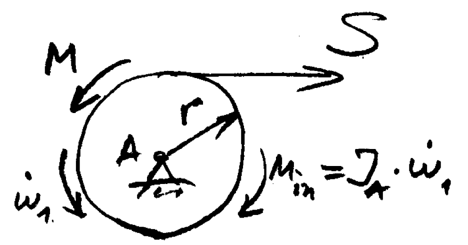
$$\longrightarrow Q_x = \frac{R-r}{R}\cdot\frac{M}{r}$$

Лагр. рав.

$$\frac{d}{dt}\frac{\partial E_k}{\partial\dot{x}} - \underbrace{\frac{\partial E_k}{\partial x}}_{0} + \underbrace{\frac{\partial E_p}{\partial x}}_{0} = Q_x\ ;\quad \frac{d}{dt}\frac{\partial E_k}{\partial\dot{x}} = 2\left[\frac{1}{4}m\left(\frac{R-r}{R}\right)^2 + m + \frac{1}{2}\frac{J_0}{R^2}\right]\ddot{x}$$

$$\left[\frac{m}{2}\left(\frac{R-r}{R}\right)^2 + 2m + \frac{J_0}{R^2}\right]\cdot\ddot{x} = \frac{R-r}{R}\cdot\frac{M}{r}\ ;\quad \boxed{\ddot{x} = \frac{\frac{R-r}{R}\cdot\frac{M}{r}}{\frac{m}{2}\left(\frac{R-r}{R}\right)^2 + 2m + \frac{J_0}{R^2}}}$$

Даламберов принцип; $\sum M_A = 0$ $\quad M - M_{in} - S\cdot r = 0$



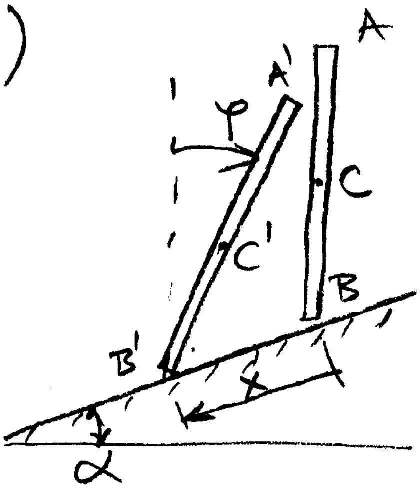
$$M - J_A\cdot\dot{\omega}_1 - S\cdot r = 0 \rightarrow S = \frac{1}{r}\left(M - J_A\cdot\dot{\omega}_1\right)$$

$$\omega_1 = \frac{\dot{x}}{r}\cdot\frac{R-r}{R}\ \Big/\frac{d}{dt} \rightarrow \dot{\omega}_1 = \frac{R-r}{r\cdot R}\cdot\ddot{x}$$

$$S = \frac{1}{r}\left[M - \frac{1}{2}mr^2\cdot\frac{R-r}{r\cdot R}\cdot\frac{\frac{R-r}{R}\cdot\frac{M}{r}}{\frac{m}{2}\left(\frac{R-r}{R}\right)^2 + 2m + \frac{J_0}{R^2}}\right]$$

$$S = \frac{M}{r}\cdot\left[1 - \frac{1}{1 + 4\left(\frac{R}{R-r}\right)^2 + \frac{2J_0}{m(R-r)^2}}\right]$$

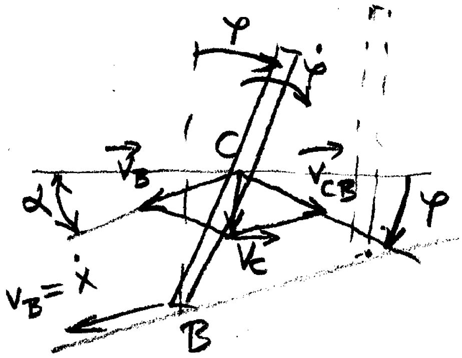
2.)



2 ст. на свобода

x, φ – генер. коорд.

$$E_K = \frac{1}{2} m \cdot v_C^2 + \frac{1}{2} J_C \cdot \dot{\varphi}^2$$



$$\vec{V}_C = \vec{V}_B + \vec{V}_{CB} \quad \Big|^2$$

$$V_C^2 = V_B^2 + 2\vec{V}_B \cdot \vec{V}_{CB} + V_{CB}^2$$

$$V_C^2 = V_B^2 + 2 V_B V_{CB} \cdot \cos(\sphericalangle \vec{V}_B, \vec{V}_{CB}) + V_{CB}^2$$

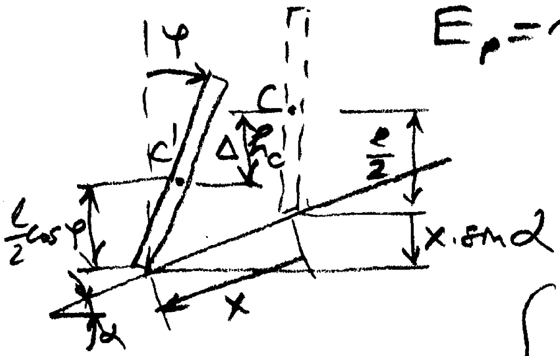
$$V_B = \dot{x}, \quad V_{CB} = \frac{\ell}{2} \cdot \dot{\varphi}$$

$$\sphericalangle \vec{V}_B, \vec{V}_{CB} = \pi - (\alpha + \varphi)$$

$$V_C^2 = \dot{x}^2 + \cancel{2} \dot{x} \cdot \frac{\ell}{\cancel{2}} \dot{\varphi} \cdot \underbrace{\cos(\pi - (\alpha+\varphi))}_{-\cos(\alpha+\varphi)} + \frac{\ell^2}{4} \cdot \dot{\varphi}^2 = \dot{x}^2 - \ell \cos(\alpha+\varphi)\dot{x}\dot{\varphi} + \frac{\ell^2}{4}\dot{\varphi}^2$$

$$E_K = \frac{1}{2} m \left(\dot{x}^2 - \ell \cos(\alpha+\varphi)\dot{x}\dot{\varphi} + \frac{\ell^2}{4}\dot{\varphi}^2\right) + \frac{1}{2} \cdot \frac{1}{12} m \ell^2 \cdot \dot{\varphi}^2$$

$$E_K = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \ell \cdot \cos(\alpha+\varphi)\dot{x}\dot{\varphi} + \frac{1}{6} m \ell^2 \dot{\varphi}^2$$



$$E_p = mg\Delta h_C; \quad \Delta h_C = -\frac{\ell}{2} - x\sin\alpha + \frac{\ell}{2}\cos\varphi = -\frac{\ell}{2}(1-\cos\varphi) - x \cdot \sin\alpha$$

$$E_p = -mg\sin\alpha \cdot x - \frac{1}{2} mg\ell(1-\cos\varphi)$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial E_k}{\partial \dot{x}} - \dfrac{\partial E_k}{\partial x} + \dfrac{\partial E_p}{\partial x} = 0 \\[2ex] \dfrac{d}{dt}\dfrac{\partial E_k}{\partial \dot{\varphi}} - \dfrac{\partial E_k}{\partial \varphi} + \dfrac{\partial E_p}{\partial \varphi} = 0 \end{cases}$$

$$\frac{\partial E_k}{\partial \dot{x}} = m\dot{x} - \frac{1}{2} m\ell \cdot \cos(\alpha+\varphi) \cdot \dot{\varphi}; \quad \frac{\partial E_k}{\partial x} = 0; \quad \frac{\partial E_p}{\partial x} = -mg\sin\alpha$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}} = m\ddot{x} + \frac{1}{2} m\ell \cdot \sin(\alpha+\varphi) \cdot \dot{\varphi}^2 - \frac{1}{2} m\ell \cdot \cos(\alpha+\varphi) \cdot \ddot{\varphi}$$

$$\frac{\partial E_k}{\partial \dot{\varphi}} = -\frac{1}{2} m\ell \cdot \cos(\alpha+\varphi) \cdot \dot{x} + \frac{1}{3} m\ell^2 \dot{\varphi}; \quad \frac{\partial E_k}{\partial \varphi} = \frac{1}{2} m\ell \cdot \sin(\alpha+\varphi)\dot{x}\dot{\varphi}$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\varphi}} = \frac{1}{2} m\ell \cdot \sin(\alpha+\varphi)\dot{x}\dot{\varphi} - \frac{1}{2} m\ell \cdot \cos(\alpha+\varphi)\ddot{x} + \frac{1}{3} m\ell^2 \ddot{\varphi}$$

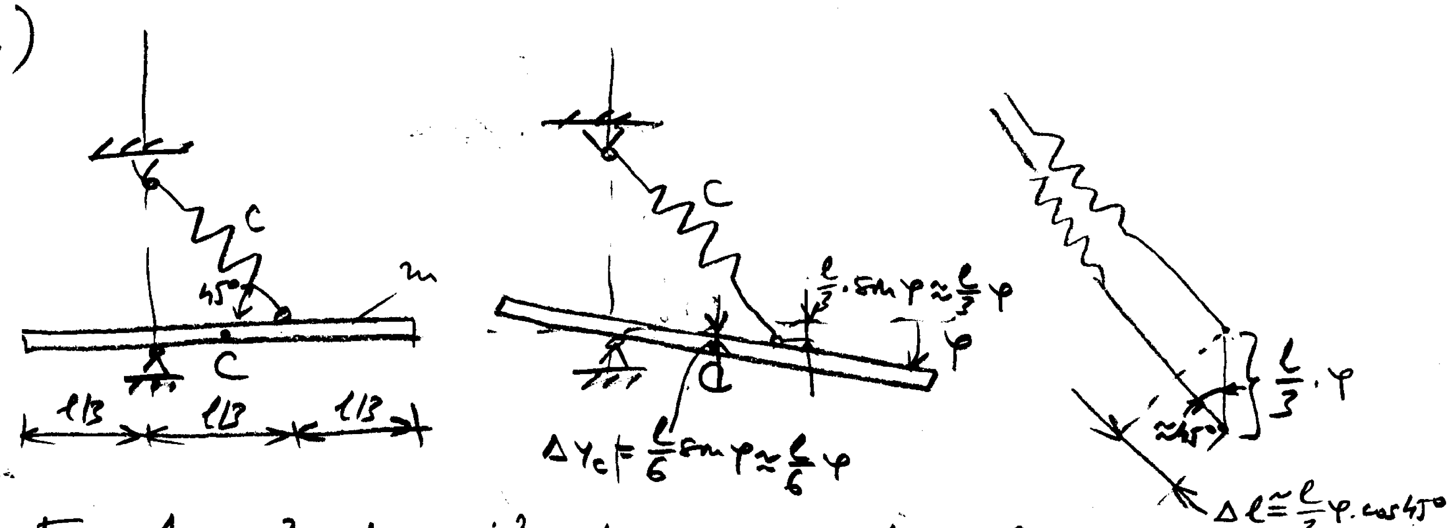
$$\frac{\partial E_p}{\partial \varphi} = -\frac{1}{2} mg\ell \cdot \sin\varphi$$

3

$$\begin{cases} m\ddot{x} - \frac{1}{2} m l \cdot \cos(\alpha+\varphi) \cdot \ddot{\varphi} + \frac{1}{2} m l \cdot \sin(\alpha+\varphi) \cdot \dot{\varphi}^2 - mg \sin\alpha = 0 \quad /:m \\ -\frac{1}{2} m l \cdot \cos(\alpha+\varphi)\, \ddot{x} + \frac{1}{3} m l^2 \ddot{\varphi} + \cancel{\frac{1}{2} m l \cdot \sin(\alpha+\varphi)\, \dot{x}\dot{\varphi}} - \cancel{\frac{1}{2} m l \sin(\alpha+\varphi)\, \dot{x}\dot{\varphi}} - \\ \qquad - \frac{1}{2} m g\, l \cdot \sin\varphi = 0 \quad /:(-\frac{1}{2} m l) \end{cases}$$

$$\begin{cases} \ddot{x} - \frac{l}{2} \cdot \cos(\alpha+\varphi) \cdot \ddot{\varphi} + \frac{l}{2} \cdot \sin(\alpha+\varphi) \cdot \dot{\varphi}^2 - g \cdot \sin\alpha = 0 & (1) \\ \cos(\alpha+\varphi)\, \ddot{x} - \frac{2}{3} l\, \ddot{\varphi} + g \cdot \sin\varphi = 0 & (2) \end{cases}$$

3.)



$$E_k = \frac{1}{2} m v_c^2 + \frac{1}{2} J_c \cdot \dot{\varphi}^2 = \frac{1}{2} m \left(\frac{l}{6}\dot{\varphi}\right)^2 + \frac{1}{2} \cdot \frac{1}{12} m l^2 \dot{\varphi}^2$$

$$\underline{E_k = \frac{1}{18} m l^2 \dot{\varphi}^2}$$

$$E_p = \frac{1}{2} c (\Delta l_{st} + \Delta l)^2 - \frac{1}{2} c \cdot \Delta l_{st}^2 - mg \frac{l}{6} \cdot \varphi = \frac{1}{2} c \left(\Delta l_{st} + \frac{\sqrt{2}}{2} \cdot \frac{l}{3} \cdot \varphi\right)^2 -$$

$$- \frac{1}{2} c \Delta l_{st}^2 - mg \frac{l}{6} \cdot \varphi = \cancel{\frac{1}{2} c \Delta l_{st}^2} + \underline{c \Delta l_{st} \cdot \frac{\sqrt{2}}{2} \cdot \frac{l}{3} \cdot \varphi} + \underline{\underline{\frac{1}{2} c \frac{2}{4} \cdot \frac{l^2}{9} \varphi^2}} -$$

$$- \cancel{\frac{1}{2} c \Delta l_{st}^2} \underline{- mg \frac{l}{6} \varphi} = \frac{1}{36} c l^2 \varphi^2 + \left(c \Delta l_{st} \frac{\sqrt{2}}{2} \cdot \frac{l}{3} - mg \frac{l}{6}\right) \cdot \varphi$$

$\to 0$ ([illegible])

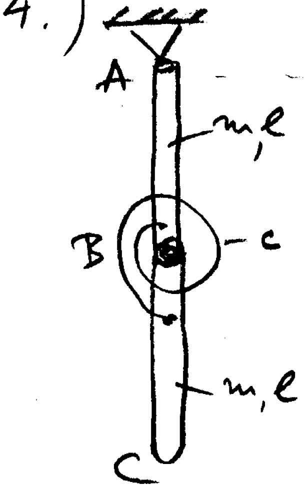
$$\left.\frac{dE_p}{d\varphi}\right|_{\varphi=0} = 0 \; ; \; \left.\frac{dE_p}{d\varphi}\right|_{\varphi=0} = \frac{1}{18} c l^2 \varphi + c \Delta l_{st} \cdot \frac{\sqrt{2}}{2} \cdot \frac{l}{3} - mg \frac{l}{6} \Bigg|_{\varphi=0} = 0$$

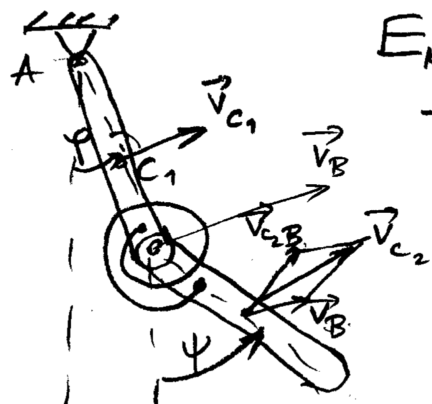
$$\to c \Delta l_{st} \cdot \frac{\sqrt{2}}{2} \cdot \frac{l}{3} - mg \frac{l}{6} = 0$$

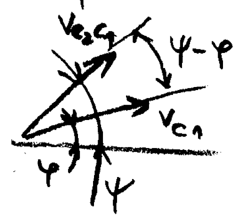
$$E_p \cong \frac{1}{36} c l^2 \varphi^2 \; ; \quad \frac{d}{dt} \frac{\partial E_k}{\partial \dot{\varphi}} - \underbrace{\frac{\partial E_k}{\partial \varphi}}_{0} + \frac{\partial E_p}{\partial \varphi} = 0$$

$$\frac{1}{9} m l^2 \ddot{\varphi} + \frac{1}{18} c l^2 \varphi = 0 \qquad \ddot{\varphi} + \underbrace{\frac{1}{2} \cdot \frac{c}{m}}_{\omega^2} \varphi = 0 \to \boxed{\omega = \sqrt{\frac{c}{2m}}}$$

4.)







$$E_K = \frac{1}{2} J_A \cdot \dot\varphi^2 + \frac{1}{2} m V_{C_2}^2 + \frac{1}{2} J_{C_2} \cdot \dot\psi^2$$

$$\vec{V}_{C_2} = \vec{V}_B + \vec{V}_{C_2B} \quad \Big|^2$$

$$V_{C_2}^2 = V_B^2 + 2 V_B \cdot V_{C_2B} \cdot \cos\sphericalangle(\vec{V}_B, \vec{V}_{C_2B}) + V_{C_2B}^2$$

$$V_B = l \cdot \dot\varphi, \quad V_{C_2C_1} = \frac{l}{2} \cdot \dot\psi$$

$$\sphericalangle \vec{V}_B, \vec{V}_{C_1B} = \psi - \varphi$$

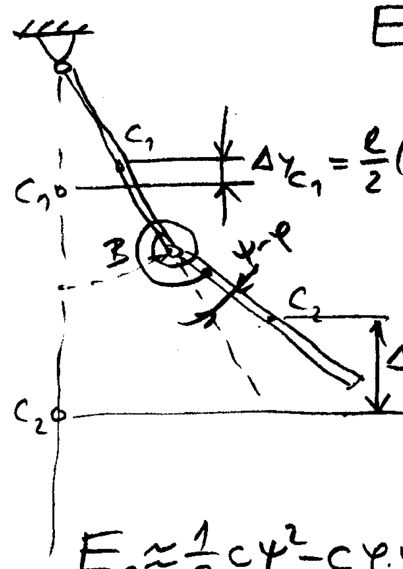
$$V_{C_2}^2 = l^2 \dot\varphi^2 + \cancel{2} l \dot\varphi \cdot \frac{l}{\cancel{2}} \dot\psi \cdot \cos(\psi-\varphi) + \frac{l^2}{4}\dot\psi^2$$

$$E_K = \frac{1}{2}\cdot\frac{1}{3} m l^2 \cdot \dot\varphi^2 + \frac{1}{2} m \Big[l^2 \dot\varphi^2 + l^2 \underbrace{\cos(\psi-\varphi)}_{\approx 1} \cdot \dot\varphi \dot\psi + \frac{l^2}{4}\dot\psi^2 \Big] + \frac{1}{2}\cdot\frac{1}{12} m l^2 \dot\psi^2$$

За мали осцилации $E_K(\dot\varphi, \dot\psi, \varphi, \psi) \approx E_K(\dot\varphi, \dot\psi, \varphi=0, \psi=0)$

$$\cos(\psi-\varphi) \approx \cos(0-0) = 1$$

$$E_K \approx \frac{2}{3} m l^2 \dot\varphi^2 + \frac{1}{2} m l^2 \dot\varphi \dot\psi + \frac{1}{6} m l^2 \dot\psi^2$$



$$\Delta Y_{C_1} = \frac{l}{2}(1-\cos\varphi)$$

$$\Delta Y_{C_2} = \Delta Y_B + \frac{l}{2}(1-\cos\psi) = l(1-\cos\varphi) + \frac{l}{2}(1-\cos\psi)$$

$$E_p = \frac{1}{2} c \cdot (\psi-\varphi)^2 + mg\,\Delta Y_{C_1} + mg\,\Delta Y_{C_2}$$

$$E_p = \frac{1}{2} c(\psi-\varphi)^2 + mg \frac{l}{2} \overbrace{(1-\cos\varphi)}^{\approx \varphi^2/2} + mg\Big[l \underbrace{(1-\cos\varphi)}_{\approx \frac{\varphi^2}{2}} + \frac{l}{2} \underbrace{(1-\cos\psi)}_{\approx \frac{\psi^2}{2}} \Big]$$

$$E_p \approx \frac{1}{2} c(\psi-\varphi)^2 + \frac{1}{4} mgl\varphi^2 + \frac{1}{2} mgl\varphi^2 + \frac{1}{4} mgl\psi^2$$

$$E_p \approx \frac{1}{2} c\psi^2 - c\varphi\cdot\psi + \frac{1}{2} c\varphi^2 + \frac{3}{4} mgl\varphi^2 + \frac{1}{4} mgl\psi^2$$

$$E_p \approx \left(\frac{3}{4} mgl + \frac{1}{2} c\right)\varphi^2 - c\varphi\cdot\psi + \left(\frac{1}{4} mgl + \frac{1}{2} c\right)\psi^2$$

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial E_K}{\partial \dot\varphi} + \dfrac{\partial E_p}{\partial \varphi} = 0 \\[2ex] \dfrac{d}{dt}\dfrac{\partial E_K}{\partial \dot\psi} + \dfrac{\partial E_p}{\partial \psi} = 0 \end{cases} \qquad \begin{cases} \dfrac{4}{3} ml^2 \ddot\varphi + \dfrac{1}{2} ml^2 \ddot\psi + \left(\dfrac{3}{2} mgl + c\right)\varphi - c\psi = 0 \\[2ex] \dfrac{1}{2} ml^2 \ddot\varphi + \dfrac{1}{3} ml^2 \ddot\psi - c\cdot\varphi + \left(\dfrac{1}{2} mgl + c\right)\psi = 0 \end{cases}$$

$\varphi = A\cos(\omega t - \alpha)$, $\psi = B\cdot\cos(\omega t - \alpha)$

(J)

$$\begin{cases} \left[\left(\frac{3}{2}mgl + c - \frac{4}{3}ml^2\omega^2\right)A + \left(-c - \frac{1}{2}ml^2\omega^2\right)B\right]\cdot\cos(\cancel{\omega t - \alpha}) = 0 \\ \left[\left(-c - \frac{1}{2}ml^2\omega^2\right)A + \left(\frac{1}{2}mgl + c - \frac{1}{3}ml^2\omega^2\right)B\right]\cos(\cancel{\omega t - \alpha}) = 0 \end{cases}$$

$/ : \cos(\omega t - \alpha)$

$/ : ml^2$

$$\begin{cases} \left(\frac{3}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{4}{3}\omega^2\right)A - \left(\frac{c}{ml^2} + \frac{1}{2}\omega^2\right)B = 0 \\ -\left(\frac{c}{ml^2} + \frac{1}{2}\omega^2\right)A + \left(\frac{1}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{1}{3}\omega^2\right)B = 0 \end{cases}$$

$$\Delta(\omega^2) = \begin{vmatrix} \frac{3}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{4}{3}\omega^2 & -\left(\frac{c}{ml^2} + \frac{1}{2}\omega^2\right) \\ -\left(\frac{c}{ml^2} + \frac{1}{2}\omega^2\right) & \frac{1}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{1}{3}\omega^2 \end{vmatrix} = 0$$

$$\Delta(\omega^2) = \left(\frac{3}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{4}{3}\omega^2\right)\cdot\left(\frac{1}{2}\frac{g}{l} + \frac{c}{ml^2} - \frac{1}{3}\omega^2\right) - \left(\frac{c}{ml^2} + \frac{1}{2}\omega^2\right)^2 = 0$$

$\rightarrow \omega^2_{1,2}$